Course deals with

- heat equation: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ 1-dim $0 \le x \le L$

  $\frac{\partial u}{\partial t} = k \nabla^2 u$ 2-dim case $(x,y) \in R$ 2-dim region

- wave equation: $\frac{\partial^2 u}{\partial t^2} = \begin{cases} c^2 \frac{\partial^2 u}{\partial x^2} & \text{1-dim} \\ c^2 \nabla^2 u & \text{2-dim} \end{cases}$

<u>Solution</u>: product solution

2-dim. case: $u(x,y,t) = \phi(x,y)\, h(t)$.

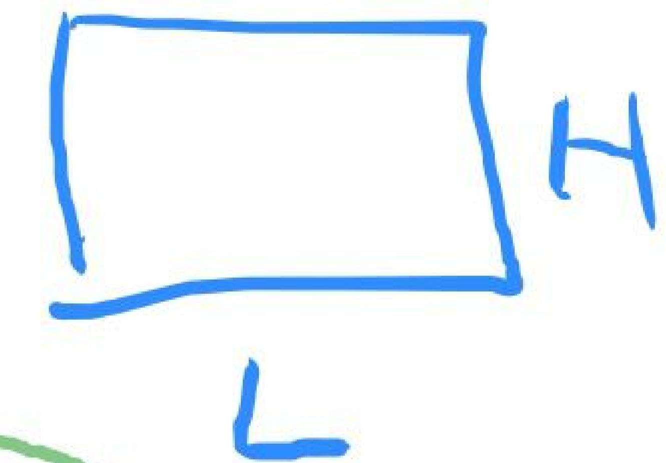
$\longrightarrow$ eigenvalue problem:

$$\boxed{\nabla^2 \phi = -\lambda \phi}$$

(same for heat & wave equation)

solution depends on shape of region R

a) R rectangle



eigenvalues: $$\boxed{\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{H^2}, \quad n=0,1,2.. \quad m=0,1,2..}$$

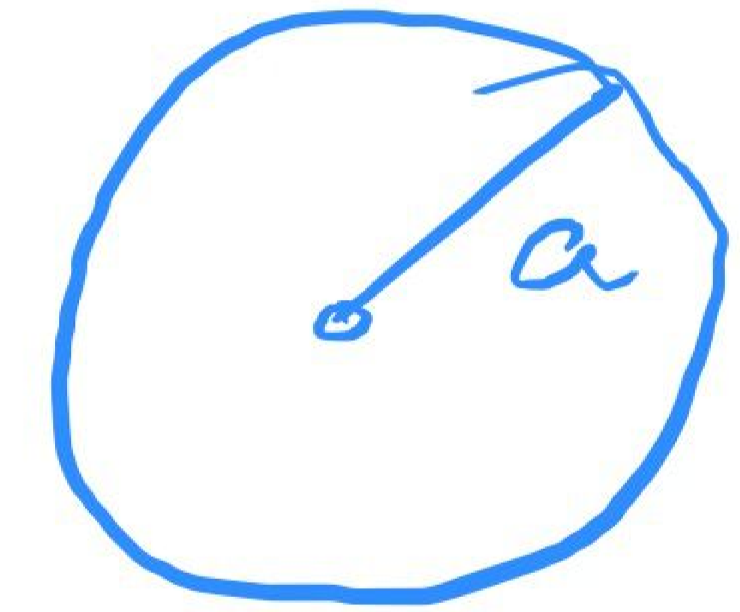
eigenfunctions: usually of form

$$\left(\begin{array}{c} \sin \frac{n\pi}{L} x \\ \text{or} \\ \cos \frac{n\pi}{L} x \end{array} \right) \cdot \left(\begin{array}{c} \sin \frac{m\pi}{H} y \\ \text{or} \\ \cos \frac{m\pi}{H} y \end{array} \right)$$

choice depends on boundary cond.
(sine or cosine)

e.g. $\sin \frac{n\pi}{L} x \cos \frac{m\pi}{H} y$

(b) $R =$ disk of radius a



use polar coordinates:

$$u(r,\theta,t) = \Phi(r,\theta)\, h(t)$$

$$= f(r)\, g(\theta)\, h(t)$$

<u>result:</u>

$$g(\theta) = a_m \cos m\theta + b_m \sin m\theta$$

$$f(r) = J_m(\sqrt{\lambda}\, r)$$

$J_m =$ m-th Bessel function

possible eigenvalues? depend on boundary cond.

e.g. if $u|_{\partial R} = 0 \Rightarrow f(a) = 0 \Rightarrow J_m(\sqrt{\lambda}\, r) = 0$

$$\Rightarrow \sqrt{\lambda}\, a = z_{m,n}$$

n-th zero of J_m

$\Rightarrow$ get eigenvalue

$$\boxed{\lambda_{mn} = \frac{z_{mn}^2}{a^2}}$$

or if bd cond.: $\frac{df}{dr} = 0 \Rightarrow \sqrt{\lambda}\, J_m'(\sqrt{\lambda} r) = 0$

$\Rightarrow$ eigenvalues given by zeros $\tilde{z}_{mn}$ of J_m'

$$\Rightarrow \boxed{\lambda_{mn} = \frac{\tilde{z}_{mn}^2}{a^2}}$$

$\Rightarrow$ general product solution:

$$\boxed{u(r,\theta,t) = \underbrace{J_m(\sqrt{\lambda_{mn}}\, r)}_{f(r)} \underbrace{\left\{ \begin{matrix} \cos m\theta \\ \sin m\theta \end{matrix} \right\}}_{g(\theta)} \underbrace{\left\{ \begin{matrix} \cos \sqrt{\lambda_{mn}}\, ct \\ \sin \sqrt{\lambda_{mn}}\, ct \end{matrix} \right\}}_{h(t)}}$$

(i.e. could be sine or cosine

② How to calculate explicit initial value problem?

heat equation: $\boxed{u(x,y,0) = \alpha(x,y)}$

wave equ.
$$\boxed{\begin{aligned} u(x,y,0) &= \alpha(x,y) \\ \frac{\partial u}{\partial t}(x,y,0) &= \beta(x,y) \end{aligned}}$$

this can be done using Fourier analysis.

e.g. for rectangle, (heat equ. for simplicity

assume bd. cond. were $u|_{\partial\Omega} = 0$

have shown: eigenfunctions $\boxed{\sin\frac{n\pi}{L}x \; \sin\frac{m\pi}{H}y}$ $n=1,2,\ldots$ $m=1,2,\ldots$

general sol. $\boxed{u(x,y,t) = \sum b_{nm} \sin\frac{n\pi}{L}x \; \sin\frac{m\pi}{H}y \; e^{-k\left(\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right)t}}$

Fourier series formula

$$\boxed{b_{mn} = \frac{2}{L}\cdot\frac{2}{H}\int_0^L\int_0^H \alpha(x,y)\sin\frac{n\pi}{L}x\,\sin\frac{m\pi}{H}y\,dy\,dx}$$

This is a general principle!

<u>Theorem:</u> Assume we have eigenfunctions ϕ_λ and ϕ_μ for same boundary conditions

(i.e. $\nabla^2\phi_\lambda = -\lambda\phi_\lambda$, $\nabla^2\phi_\mu = -\mu\phi_\mu$)

and $\lambda \neq \mu$

$$\Longrightarrow \boxed{\iint_D \phi_\lambda(x,y)\,\phi_\mu(x,y)\,dx\,dy = 0}$$

e.g. $\lambda = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$

$$\mu = \left(\frac{\tilde{n}\pi}{L}\right)^2 + \left(\frac{\tilde{m}\pi}{H}\right)^2$$

$$\Rightarrow \int_0^L\int_0^H \underbrace{\left(\sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y\right)}_{=\phi_i}\left(\sin\frac{\tilde{n}\pi}{L}x \sin\frac{\tilde{m}\pi}{H}y\right)dx\,dy = 0$$

if $n \neq \tilde{n}$ or $m \neq \tilde{m}$

If Ω = dish:

say we have eigenfunctions fa

$\lambda_{mn} = J_m(\sqrt{\lambda_{mn}}\,r)\sin m\theta$

$\lambda^{\#}_{m\tilde{n}} = J_m(\sqrt{\lambda_{m\tilde{n}}}\,r)\sin m\theta$

$$0 = \int_0^{2\pi}\int_0^a J_m(\sqrt{\lambda_{mn}}\,r)\sin m\theta \; J_m(\sqrt{\lambda_{m\tilde{n}}}\,r)\sin m\theta\, r\,dr\,d\theta =$$

$$= \underbrace{\left(\int_0^{2\pi} \sin^2 m\theta \, d\theta \right)}_{= \pi \neq 0} \left(\int_0^a J_m(\sqrt{\lambda_{mn}}\, r) \, J_m(\sqrt{\lambda_{m\tilde{n}}}\, r) \, r \, dr \right) = 0$$

orthogonality relations for Bessel functions

$$\int_0^a J_m(\sqrt{\lambda_{mn}}\, r) \, J_m(\sqrt{\lambda_{m\tilde{n}}}\, r) \, r \, dr = 0 \quad \text{if } n \neq \tilde{n}$$

$\Rightarrow$ can use this to calculate coeff. e.g.

if $\quad \alpha(r) = \sum a_n J_m(\sqrt{\lambda_{mn}}\, r) \qquad$ m fixed (e.g. $m=5$)

then we can calculate the coefficients a_n via the formula

$$\boxed{a_n = \frac{\int_0^a \alpha(r)\, J_m(\sqrt{\lambda_{mn}}\, r)\, r\, dr}{\int_0^a J_m^2(\sqrt{\lambda_{mn}}\, r)\, r\, dr}}$$

Other possible material:

Problems about convergence of Fourier Series (see 2nd midterm and problem in practice final)